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Colored noise in a two-dimensional nonlinear system

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Abstract. A two-dimensional decoupling theory is developed when colored noise is included in a nonlinear dynamical system. By a functional analysis, the colored noise is transformed to an effective noise that includes the noise correlation time, the mean dynamical variable, and the original noise strength. When the two-dimensional decoupling theory is applied to single-mode and two-mode dye laser systems, the mean, variance, and effective eigenvalue of laser intensity are calculated. Excellent agreement between theoretical analysis, numerical simulations, and experimental measurements are obtained. It is seen that the increase of noise correlation time can reduce the fluctuations in the laser system. It is also shown that there is relatively large fluctuation in the phase when the laser undergoes from thermal light to coherent light when the theory is applied to a single mode dye laser.

PACS. 05.40.Ca Noise – 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps

1 Introduction

Nonlinear dynamical systems with random noise have been paid much attention both theoretically and experimentally [1–27]. Phenomena such as noise-induced transitions [1,2], stochastic resonance [3,4], resonant activation [5,6], noise-induced spatial patterns [8,9] are a few examples of extensive investigations. The effects of colored noise in non-equilibrium systems have attracted a great deal of interests for many years [10–27]. Some of the original studies made use of two-dimensional phase measurements and Fokker-Planck theories with application of them [13–18]. Realistically, noise appears usually not only in one-dimensional systems but also in high-dimensional systems.

In this paper, a two-dimensional decoupling theory is developed when multiplicative colored noise is included in a nonlinear system. In Section 2, a two-dimensional Langevin equation with both colored and white noises is presented. By a functional analysis, the colored noise is approximately by an effective white noise. In Section 3, the two-dimensional decoupling theory is applied to a single-mode dye laser system. The mean, variance, and effective eigenvalue of the laser intensity are calculated analytically and compared to numerical simulations and experimental measurements [22,23]. In Section 4, the variance of the phase and the power spectrum of laser field are calculated. The effects of colored noise correlation time on the variance of laser intensity, phase and power spectrum are discussed. In Section 5, the two-dimensional decoupling theory is applied to a two-mode dye laser system. The mean, variance and effective eigenvalue of the laser intensity are calculated analytically and compared to experimental measurements [22,23]. A discussion of the theory concludes the paper.

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2 Two-dimensional decoupling theory

A two-dimensional stochastic system with both colored and white noises follows the Langevin equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot \boldsymbol{\xi}(t) + \mathbf{G}(\mathbf{x}) \cdot \boldsymbol{\eta}(t) \tag{1}$$

where $\mathbf{h}(\mathbf{x})$ is the deterministic part, $\mathbf{g}(\mathbf{x})$ and $\mathbf{G}(\mathbf{x})$ are diagonal matrices, $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t)$ are colored and white noise. The noise terms are statistically independent and their mean and variance are given by

$$\begin{aligned} \langle \boldsymbol{\xi}(t) \rangle &= \langle \boldsymbol{\eta}(t) \rangle = 0 \\ \langle \boldsymbol{\xi}(t) \cdot \boldsymbol{\xi}(t') \rangle &= (b/\tau) \exp(-|t-t'|/\tau) \\ \langle \boldsymbol{\eta}(t) \cdot \boldsymbol{\eta}(t') \rangle &= 2c\delta(t-t') \end{aligned} \tag{2}$$

where b and c are the colored and white noise strengths, τ is the noise correlation time. Equation (1) can also be explicitly written as

$$\begin{bmatrix} \mathrm{d}x_1/\mathrm{d}t \\ \mathrm{d}x_2/\mathrm{d}t \end{bmatrix} = \begin{bmatrix} h_1(\mathbf{x}) \\ h_2(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} g_{11}(\mathbf{x}) & 0 \\ 0 & g_{22}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + \begin{bmatrix} G_{11}(\mathbf{x}) & 0 \\ 0 & G_{22}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}. \tag{3}$$

Since the random force $\boldsymbol{\xi}(t)$ is NOT a white noise but a colored noise with finite correlation time τ, equation (3) is no longer a Markov process. Though the corresponding Fokker-Planck equation of the probability density $Q(\mathbf{x},t)$ of the variable $\mathbf{x}$ can be written in the form of [28]

$$\frac{\partial Q(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left\{h_i(\mathbf{x})Q(\mathbf{x},t) - \frac{1}{2}g_{ii}(\mathbf{x}) \times \langle \xi_i(t)\,\delta[\mathbf{x}(t)-\mathbf{x}]\rangle - \frac{c}{2}G_{ii}(\mathbf{x})\frac{\partial}{\partial x_i}[G_{ii}(\mathbf{x})Q(\mathbf{x},t)]\right\} \quad (4)$$

it is difficult to solve the above equation since the colored noise term $\boldsymbol{\xi}(t)$ involved in equation (4). By introducing the approximation method of the decoupling theory [19,20], the non-Markov process can be reduced to Markov process. The main idea of decoupling theory is to transform the colored noise $\boldsymbol{\xi}(t)$ to an effective white noise. The intensity of the effective noise includes the noise correlation time τ, the mean of the dynamical variable $\mathbf{x}$ and the original noise strength b. By a functional analysis, one has [19,20]

$$\begin{aligned}\langle \xi_i(t)\,\delta[\mathbf{x}(t)-\mathbf{x}]\rangle &= \int_0^t \mathrm{d}t' \langle \boldsymbol{\xi}(t)\cdot\boldsymbol{\xi}(t')\rangle \left\langle \frac{\delta\{\delta[\mathbf{x}(t)-\mathbf{x}]\}}{\delta\xi_i}\right\rangle \\
&= \frac{b}{\tau}\int_0^t \mathrm{d}t' \exp[-|t-t'|/\tau]\left\langle \delta[\mathbf{x}(t)-\mathbf{x}]\frac{\delta\mathbf{x}(t)}{\delta\xi_i}\right\rangle \\
&= \frac{b}{\tau}\frac{\partial}{\partial x_i}\Bigg(g_{ii}(\mathbf{x})\int_0^t \mathrm{d}t' \exp\left[-\frac{|t-t'|}{\tau}\right] \\
&\quad\times \Bigg\langle \delta[\mathbf{x}(t)-\mathbf{x}]\exp\Bigg\{\sum_{j=1}^{2}\int_{t'}^{t}\mathrm{d}s\Big[h_j^{(j)}(\mathbf{x}(s)) \\
&\quad + g_{jj}^{(j)}(\mathbf{x}(s))\,\xi_j(s) + G_{jj}^{(j)}(\mathbf{x}(s))\,\eta_j(s) - \frac{|\mathbf{g}(\mathbf{x}(s))|^{(j)}}{|\mathbf{g}(\mathbf{x}(s))|} \\
&\quad\times [h_j(\mathbf{x}(s)) + g_{jj}(\mathbf{x}(s))\,\xi_j(s) + G_{jj}(\mathbf{x}(s))\,\eta_j(s)]\Big]\Bigg\}\Bigg\rangle\Bigg) \\
&\approx \frac{b}{\tau}\frac{\partial}{\partial x_i}\Bigg(g_{ii}(\mathbf{x})\int_0^t \mathrm{d}t' \exp\left[-\frac{|t-t'|}{\tau}\right]\langle\delta[\mathbf{x}(t)-\mathbf{x}]\rangle \\
&\quad\times \exp\left\{(t-t')\sum_{j=1}^{2}\left[\left\langle h_j^{(j)}\right\rangle - \left\langle \frac{|\mathbf{g}(\mathbf{x})|^{(j)}}{|\mathbf{g}(\mathbf{x})|}h_j(\mathbf{x})\right\rangle\right]\right\}\Bigg) \\
&= b_{\text{eff}}\frac{\partial}{\partial x_i}[g_{ii}(\mathbf{x})Q(\mathbf{x},t)]\end{aligned} \quad (5)$$

where $Q(\mathbf{x},t) = \langle\delta[\mathbf{x}(t)-\mathbf{x}]\rangle$ and all the terms multiplied by $\xi_i(t)$ and $\eta_i(t)$ have been neglected. The effective colored noise strength b_{eff} is given by

$$b_{\text{eff}} = \frac{b}{1-\tau\sum_{j=1}^{2}\left[\left\langle h_j^{(j)}(\mathbf{x})\right\rangle - \left\langle \frac{|\mathbf{g}(\mathbf{x})|^{(j)}}{|\mathbf{g}(\mathbf{x})|}h_j(\mathbf{x})\right\rangle\right]} \quad (6)$$

with

$$h_j^{(j)}(\mathbf{x}) = \frac{\partial h_j(x)}{\partial x_j}, \text{ and } |\mathbf{g}(\mathbf{x})|^{(j)} = \frac{\partial}{\partial x_j}[g_{11}(\mathbf{x})\,g_{22}(\mathbf{x})] \quad (7)$$

and the angular brackets $\langle\cdots\rangle$ denotes the steady state average value of the quantity.

Thus the Fokker-Planck equation (4) can be transformed to

$$\begin{aligned}\frac{\partial Q(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\Bigg\{&h_i(\mathbf{x})Q(\mathbf{x},t) \\
&- \frac{b_{\text{eff}}}{2}g_{ii}(\mathbf{x})\frac{\partial}{\partial x_i}[g_{ii}(\mathbf{x})Q(\mathbf{x},t)] \\
&- \frac{c}{2}G_{ii}(\mathbf{x})\frac{\partial}{\partial x_i}[G_{ii}(\mathbf{x})Q(\mathbf{x},t)]\Bigg\}\end{aligned} \quad (8)$$

where b_{eff} is given by equation (6).

3 Application to single mode dye laser

The complex electric field E of a single mode dye laser with both colored and white noises and full saturation effects follows the Langevin equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \left(-K + \frac{F}{1+A|E|^2/F}\right)E + Ep(t) + q(t) \quad (9)$$

where K is the cavity decay constant, F is the gain parameter with $F = K + a_0$, a_0 is the pump parameter, A is the self saturation coefficient. The multiplicative colored noise $p(t)$ and additive white noise $q(t)$ are statistically independent and their mean and variance are given by

$$\begin{aligned}\langle p(t)\rangle &= \langle q(t)\rangle = 0 \\
\langle p^*(t)\,p(t')\rangle &= (P'/\tau)\exp\left(-\frac{|t-t'|}{\tau}\right) \\
\langle q^*(t)\,q(t')\rangle &= 2P\delta(t-t')\end{aligned} \quad (10)$$

where P' and P are the strength of colored and white noises, τ is the colored noise correlation time.

In the polar coordinates with $E = r\mathrm{e}^{\mathrm{i}\theta}$, $p = p_1 + \mathrm{i}p_2$, and $q = q_1 + \mathrm{i}q_2$, equation (9) is stochastically equivalent to the following Langevin equation

$$\begin{bmatrix}\frac{\mathrm{d}r}{\mathrm{d}t}\\ \frac{\mathrm{d}\theta}{\mathrm{d}t}\end{bmatrix} = \begin{bmatrix}\left(-K+\frac{F}{\beta}\right)r + \frac{P}{2r}\\ 0\end{bmatrix} + \begin{bmatrix}r & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & \frac{1}{r}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix} \quad (11)$$

with $\beta = 1 + (Ar^2/F)$.

According to equations (6, 7), the effective colored noise strength in a single mode dye laser is given by

$$P'_{\text{eff}} = \frac{P'\langle I\rangle}{\tau P + \langle I\rangle\left(1 + 2\tau A\langle I\rangle/\beta_{\text{s}}^2\right)} \quad (12)$$

where

$$\langle I\rangle = \langle r^2\rangle, \ \beta_s = 1 + A\langle I\rangle/F. \tag{13}$$

Then the corresponding Fokker-Planck equation of the probability density function $Q(r,\theta;t)$ can be written as

$$\begin{aligned}\frac{\partial Q(r,\theta;t)}{\partial t} = &-\frac{\partial}{\partial r}\left\{\left[\left(-K+\frac{F}{\beta}\right)+\frac{1}{2r^2}\left(P+P'_{\rm eff}r^2\right)\right]\right.\\ &\left.\times rQ(r,\theta;t)-\frac{1}{2}\frac{\partial}{\partial r}\left(P+P'_{\rm eff}r^2\right)Q(r,\theta;t)\right\}\\ &+\frac{1}{2r^2}\left(P+P'_{\rm eff}r^2\right)\frac{\partial^2 Q(r,\theta;t)}{\partial\theta^2}\end{aligned} \tag{14}$$

where $P'_{\rm eff}$ is given by equation (12).

If the probability density function $Q(r,\theta;t)$ can be written as $Q(r,t)\,\Phi(\theta,t)$, equation (14) can be separated into two parts of amplitude r and phase θ as follows

$$\begin{aligned}\frac{\partial Q(r,t)}{\partial t} = &-\frac{\partial}{\partial r}\left\{\left[\left(-K+\frac{F}{\beta}\right)+\frac{1}{2r^2}\left(P+P'_{\rm eff}r^2\right)\right]\right.\\ &\left.\times rQ(r,t)-\frac{1}{2}\frac{\partial}{\partial r}\left[\left(P+P'_{\rm eff}r^2\right)Q(r,t)\right]\right\}\end{aligned} \tag{15}$$

and

$$\frac{\partial\Phi(\theta,t)}{\partial t}\approx\frac{1}{2}\left(\frac{P}{\langle I\rangle}+P'_{\rm eff}\right)\frac{\partial^2\Phi(\theta,t)}{\partial\theta^2} \tag{16}$$

where P/r^2 is approximated by $P/\langle I\rangle$ with $I=r^2$.

By a straightforward calculation, the steady state solution of equation (15) is given by

$$Q_s(r) = N_0 r\left(1+\frac{P'_{\rm eff}r^2}{P}\right)^{\beta_0-\alpha_0}\left(1+\frac{Ar^2}{F}\right)^{-\beta_0} \tag{17}$$

where N_0 is the normalization constant, and

$$\alpha_0 = \frac{K}{P'_{\rm eff}}+1, \ \beta_0 = \frac{F^2}{(FP'_{\rm eff}-AP)}\cdot \tag{18}$$

Then the mean $\langle I\rangle$, variance $\lambda_{2I}(0)$ and the effective eigenvalue $\lambda_{\rm eff}$ of the steady state laser intensity can be easily calculated from equation (17) with [23]

$$\langle I^n\rangle = \int_0^\infty r^{2n}Q_s(r)\,{\rm d}r, \tag{19}$$

$$\lambda_{2I}(0) = \frac{\langle I^2\rangle}{\langle I\rangle^2}-1 \tag{20}$$

and

$$\lambda_{\rm eff} = \frac{2P\langle I\rangle}{\langle I^2\rangle-\langle I\rangle^2}\cdot \tag{21}$$

To match the experimental measurements and check the accuracy of the decoupling theory, numerical simulation is performed by integrating the differential equations (9, 10). The stationary distribution function $Q_s(r)$ is constructed and the variance of the intensity $\lambda_{2I}(0)$ and the effective eigenvalue $\lambda_{\rm eff}$ are calculated numerically.

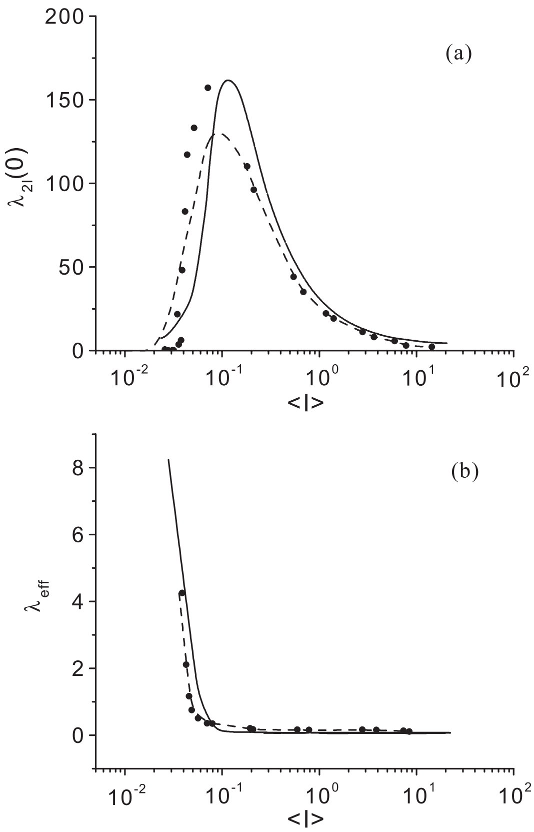

Fig. 1. The intensity variance $\lambda_{2I}(0)$ and effective eigenvalue $\lambda_{\rm eff}$ of the single-mode dye laser as a function of the average intensity $\langle I\rangle$ where $\lambda_{2I}(0)$, $\lambda_{\rm eff}$, and $\langle I\rangle$ are dimensionless. The parameters are obtained from references [22,23] and in dimensionless unit: $K=5000$, $P'=300$, $P=1$, $A=1$, $\tau=0.2$. (• • •) Experimental measurements [22,23]; (– – –) numerical simulations; (—) predictions of decoupling theory of equations (20, 21). (a) $\lambda_{2I}(0)$; (b) $\lambda_{\rm eff}$.

The theoretical predictions of equations (20, 21), the numerical simulations of equations (9, 10), together with experimental measurements from references [22,23] of the variance $\lambda_{2I}(0)$ and effective eigenvalue $\lambda_{\rm eff}$ of the mean dye laser intensity $\langle I\rangle$ are compared in Figure 1. The results from decoupling theory give slightly lower values in $\lambda_{2I}(0)$ and slightly higher values in $\lambda_{\rm eff}$ when the laser is operated well below threshold. When the laser is operated near to above threshold, the results from decoupling theory, numerical simulations and experimental measurements are almost identical. It is clear that very good agreement between theoretical analysis of the decoupling theory, numerical simulations and experimental measurements is obtained.

4 Effects of noise correlation time

The effects of non-zero noise correlation time τ on the intensity and phase fluctuations can be quantitatively investigated through equations (16, 17).

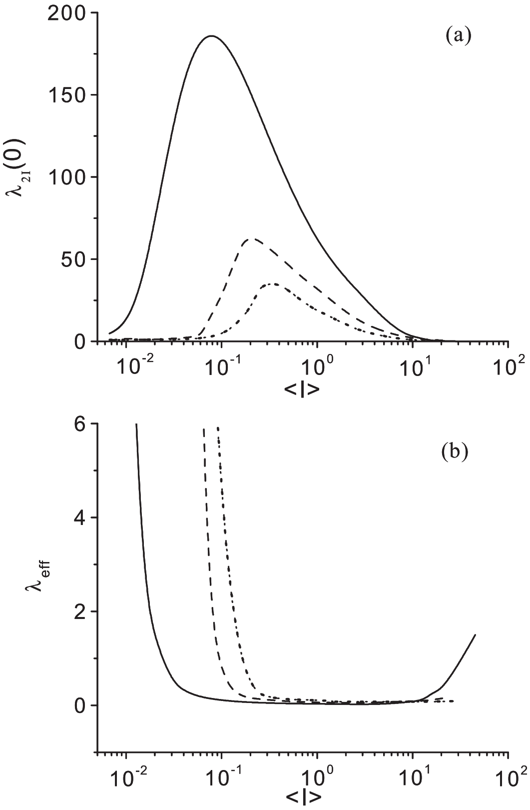

Fig. 2. The intensity variance $\lambda_{2I}(0)$ and effective eigenvalue $\lambda_{\rm eff}$ of the single-mode dye laser as a function of the average intensity $\langle I\rangle$ for different values of the noise correlation time τ where $\lambda_{2I}(0)$, $\lambda_{\rm eff}$, $\langle I\rangle$, and τ are dimensionless. The parameters are dimensionless and chosen as follows: $K = 1000$, $P' = 100$, $P = 1$, $A = 1$. (—) $\tau = 0.0$; (− − −) $\tau = 0.2$; (− · −) $\tau = 0.5$. (a) $\lambda_{2I}(0)$; (b) $\lambda_{\rm eff}$.

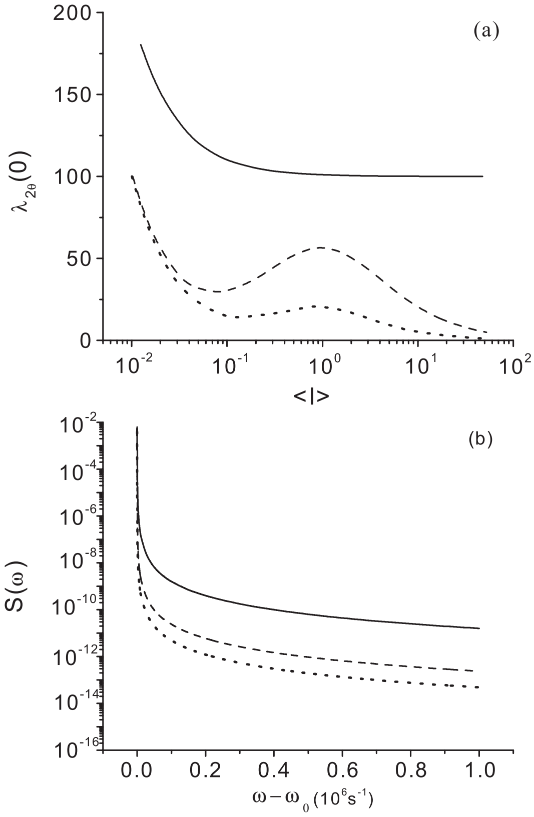

Fig. 3. The phase variance $\lambda_{2\theta}(0)$ and the power spectrum $S(\omega)$ of the single-mode dye laser field for different values of the noise correlation time τ where $\lambda_{2\theta}(0)$, $S(\omega)$, and τ are dimensionless. The parameters are dimensionless and chosen as follows: $K = 1000$, $P' = 100$, $P = 1$, $A = 1$. (—) $\tau = 0.0$; (− − −) $\tau = 0.2$; (· · ·) $\tau = 1.0$. (a) $\lambda_{2\theta}(0)$ with $t = 1.0$; (b) $S(\omega)$ with $a_0 = 200$.

4.1 Intensity fluctuation

The intensity fluctuation $\lambda_{2I}(0)$ and the effective eigenvalue $\lambda_{\rm eff}$ as a function of the mean laser intensity $\langle I\rangle$ are plotted in Figure 2 for different values of the noise correlation time τ. From Figure 2a, it is clear that the intensity variance $\lambda_{2I}(0)$ decreases as the noise correlation time τ increases. The peak in $\lambda_{2I}(0)$ is shifted from well below threshold to near threshold as τ increases. From Figure 2b, it is seen that the curve of the effective eigenvalue $\lambda_{\rm eff}$ is like "U" shape when the multiplicative noise is white with correlation time $\tau = 0$. While for non-zero τ, the curve of $\lambda_{\rm eff}$ is like "L" shape. The effective eigenvalue $\lambda_{\rm eff}$ decreases quickly when the laser is operated well below threshold. The curve of $\lambda_{\rm eff}$ becomes flat with a long flat tail when laser is operated well above threshold for non-zero values of τ. It is obvious that increasing the value of the noise correlation time τ can reduce the fluctuation in the dye laser system. This means that the noise color can suppress the fluctuations in nonlinear dynamical system.

4.2 Phase diffusion

The effects of colored noise on the phase fluctuation of a dye laser can also be investigated through equation (16). Equation (16) is a one-dimensional diffusion equation with a solution [29]

$$\Phi(\theta,t) = \sqrt{\frac{\langle I\rangle}{2\pi\left(P+P'_{\rm eff}\langle I\rangle\right)t}}\exp\left[-\frac{\theta^2\langle I\rangle}{2\left(P+P'_{\rm eff}\langle I\rangle\right)t}\right]. \tag{22}$$

The variance of the phase variable θ and the power spectrum of the dye laser field can be calculated from equation (22). The variance $\lambda_{2\theta}(0)$ of the phase θ is given by

$$\lambda_{2\theta}(0) = \left\langle\theta^2\right\rangle - \left\langle\theta\right\rangle^2 = \left(\frac{P}{\langle I\rangle} + P'_{\rm eff}\right)t. \tag{23}$$

The power spectrum $S(\omega)$ is given by

$$S(\omega) = \frac{P'_{\rm eff} + P/\langle I\rangle}{2\pi\left[(\omega-\omega_0)^2 + \frac{1}{4}\left(\frac{P}{\langle I\rangle} + P'_{\rm eff}\right)^2\right]} \tag{24}$$

where $P'_{\rm eff}$ is given by equation (12).

The variance $\lambda_{2\theta}(0)$ of the phase and the power spectrum $S(\omega)$ of the dye laser field are plotted in Figure 3 as a function of the mean laser intensity $\langle I\rangle$ for different values of noise correlation time τ.

From Figure 3a, it is seen that $\lambda_{2\theta}(0)$ decreases monotonically with a long flat tail as $\langle I\rangle$ increases for white multiplicative noise with $\tau = 0$. When τ increases, $\lambda_{2\theta}(0)$ decreases for small values of $\langle I\rangle$. After it reaches a minimum value, $\lambda_{2\theta}(0)$ increases to a peak and then decreases to zero as $\langle I\rangle$ increases further. The height between the minimum and maximum in $\lambda_{2\theta}(0)$ decreases as τ increases. The peak in $\lambda_{2\theta}(0)$ is located near laser threshold. This means that large fluctuation appears in the phase when the laser undergoes from thermal light to coherent light. When $\langle I\rangle$ increases further, $\lambda_{2\theta}(0)$ tends to zero. This means that the phase fluctuation dies out for very large laser intensity $\langle I\rangle$.

Figure 3b is a plot of the power spectrum $S(\omega)$. It is seen that $S(\omega)$ is Lorentzian shape and decreases monotonically as the angular frequency $(\omega - \omega_0)$ increases. The power spectrum $S(\omega)$ decreases as the noise correlation time τ increases.

5 Application to two-mode dye laser

It is not surprising that the two-dimensional decoupling theory can be successfully applied to a single-mode dye laser and predict the phase properties of the laser, since the one-dimensional decoupling theory already gave excellent results for the fluctuation of a single mode dye laser intensity [24–27]. However, when the two-dimensional decoupling theory is applied to a two-mode dye laser system, it can give very nice theoretical predictions for the laser intensity auto-correlation function and effective eigenvalue.

The complex electric fields E_1 and E_2 of a two-mode dye laser with both colored and white noise follow the Langevin equation

$$\begin{aligned}\frac{\mathrm{d}E_1}{\mathrm{d}t} &= a_1E_1 - A|E_1|^2E_1 - \xi|E_2|^2E_1 + E_1p(t) + q_1(t)\\ \frac{\mathrm{d}E_2}{\mathrm{d}t} &= a_2E_2 - A|E_2|^2E_2 - \xi|E_1|^2E_2 + E_2p(t) + q_2(t)\end{aligned} \tag{25}$$

where $\xi = 2$ is the mode coupling constant for the two-mode dye laser. The multiplicative colored noise $p(t)$ and additive white noise $q_i(t)$ are statistically independent and their mean and variance are given by

$$\begin{aligned}&\langle p(t)\rangle = \langle q_i(t)\rangle = 0\\ &\langle p^*(t)\,p(t')\rangle = (P'/\tau)\exp\left(-\frac{|t-t'|}{\tau}\right) \quad (i,j=1,2)\\ &\langle q_i^*(t)\,q_j(t)\rangle = 2P\delta_{ij}\delta(t-t')\end{aligned} \tag{26}$$

where P', P and τ have the same meaning as that in equation (10).

In the polar coordinates with $E_1 = r_1\mathrm{e}^{\mathrm{i}\phi_1}$, $E_2 = r_2\mathrm{e}^{\mathrm{i}\phi_2}$, $p = p_r + \mathrm{i}p_i$, $q_1 = q_{1r} + \mathrm{i}q_{1i}$, and $q_2 = q_{2r} + \mathrm{i}q_{2i}$, equation (25) is stochastically equivalent to the following Langevin equations

$$\begin{bmatrix}\frac{\mathrm{d}r_1}{\mathrm{d}t}\\ \frac{\mathrm{d}r_2}{\mathrm{d}t}\end{bmatrix} = \begin{bmatrix}a_1r_1 - Ar_1^3 - \xi r_2^2r_1 + \frac{P}{2r_1}\\ a_2r_2 - Ar_2^3 - \xi r_1^2r_2 + \frac{P}{2r_2}\end{bmatrix} + \begin{bmatrix}r_1 & 0\\ 0 & r_2\end{bmatrix}\begin{bmatrix}p_{1r}\\ p_{2r}\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}q_{1r}\\ q_{2r}\end{bmatrix}. \tag{27}$$

According to equations (6, 7), the effective colored noise strength in a two-mode dye laser is

$$P'_{\mathrm{eff}} = \frac{\langle I_1\rangle\langle I_2\rangle P'}{\langle I_1\rangle\langle I_2\rangle + \tau\left(2A\langle I_1\rangle\langle I_2\rangle + P\right)\left(\langle I_1\rangle + \langle I_2\rangle\right)} \tag{28}$$

where $\langle I_1\rangle = \langle r_1^2\rangle$, and $\langle I_2\rangle = \langle r_2^2\rangle$.

Then the corresponding Fokker-Planck equation of the probability density function $Q(r_1, r_2; t)$ can be written as [28]

$$\begin{aligned}\frac{\partial Q}{\partial t} = &-\frac{\partial}{\partial r_1}\left\{\left(a_1r_1 - Ar_1^3 - \xi r_2^2r_1 + \frac{P}{2r_1} + \frac{P'_{\mathrm{eff}}}{2}r_1\right)Q\right.\\ &\left.-\frac{\partial}{\partial r_1}\left[\frac{1}{2}\left(P + P'_{\mathrm{eff}}r_1^2\right)Q\right]\right\}\\ &-\frac{\partial}{\partial r_2}\left\{\left(a_2r_2 - Ar_2^3 - \xi r_1^2r_2 + \frac{P}{2r_2} + \frac{P'_{\mathrm{eff}}}{2}r_2\right)Q\right.\\ &\left.-\frac{\partial}{\partial r_2}\left[\frac{1}{2}\left(P + P'_{\mathrm{eff}}r_2^2\right)Q\right]\right\}.\end{aligned} \tag{29}$$

For time $t \to \infty$, the system reaches the steady state. Then equation (29) is reduced to $\partial Q(r_1, r_2; t)/\partial t = 0$. The steady state distribution function $Q(r_1, r_2)$ can be calculated directly if the mean field theory is adopted with $\xi r_2^2r_1 \approx \xi\langle r_2^2\rangle r_1$ and $\xi r_1^2r_2 \approx \xi\langle r_1^2\rangle r_2$ in equation (29). Thus one has

$$Q(r_1, r_2) = Nr_1r_2\left(P + P'_{\mathrm{eff}}r_1^2\right)^{\beta_1}\left(P + P'_{\mathrm{eff}}r_2^2\right)^{\beta_2} \times \exp\left[-\alpha_1\left(r_1^2 + r_2^2\right)\right] \tag{30}$$

where N is the normalization constant, and

$$\begin{aligned}\alpha_1 &= A/P'_{\mathrm{eff}}\\ \beta_1 &= \left(a_1 - \xi\langle r_2^2\rangle + AP/P'_{\mathrm{eff}}\right)/P'_{\mathrm{eff}} - 1\\ \beta_2 &= \left(a_2 - \xi\langle r_1^2\rangle + AP/P'_{\mathrm{eff}}\right)/P'_{\mathrm{eff}} - 1.\end{aligned} \tag{31}$$

Then the mean $\langle I_1\rangle, \langle I_2\rangle$, variance $\lambda_{11}(0)$ and the effective eigenvalue $\lambda_{\mathrm{eff}}^{11}$ of the steady state laser intensity can be easily calculated from equation (30) with [23]

$$\langle I_i^n\rangle = \int_0^\infty r_i^{2n}Q_{\mathrm{s}}(r_1, r_2)\,\mathrm{d}r_1\mathrm{d}r_2 \quad (i,j=1,2) \tag{32}$$

$$\lambda_{11}(0) = \langle I_1^2\rangle/\langle I_1\rangle^2 - 1 \tag{33}$$

and

$$\lambda_{\mathrm{eff}}^{11} = \frac{2P\langle I_1\rangle}{\langle I_1^2\rangle - \langle I_1\rangle^2}. \tag{34}$$

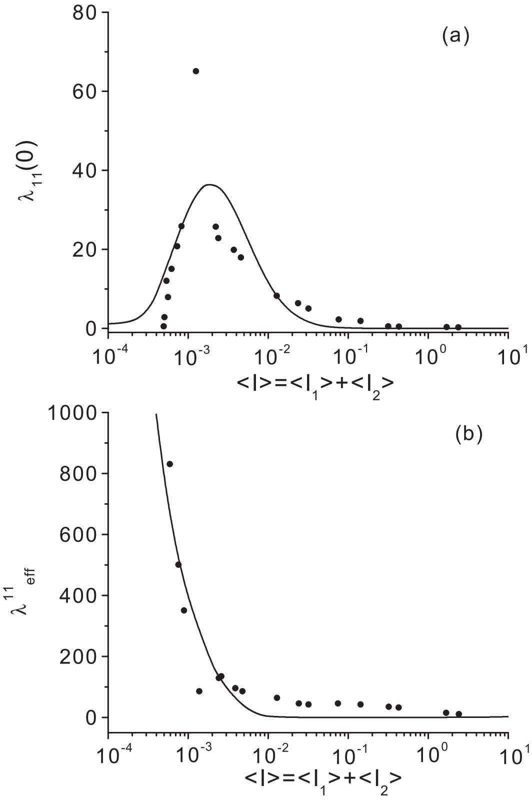

Fig. 4. The intensity variance $\lambda_{11}(0)$ and effective eigenvalue $\lambda^{11}_{\text{eff}}$ of the two-mode dye laser as a function of the average intensity $\langle I\rangle = \langle I_1\rangle + \langle I_2\rangle$ where $\lambda_{11}(0)$, $\lambda^{11}_{\text{eff}}$, and $\langle I\rangle$ are dimensionless. The parameters are obtained from references [22,23] and in dimensionless unit: $P' = 240$, $P = 1$, $A = 1$, $\tau = 0.2$. ($\bullet\bullet\bullet$) experimental measurements [22,23]; (—) predictions of decoupling theory of equations (33, 34). (a) $\lambda_{11}(0)$; (b) $\lambda^{11}_{\text{eff}}$.

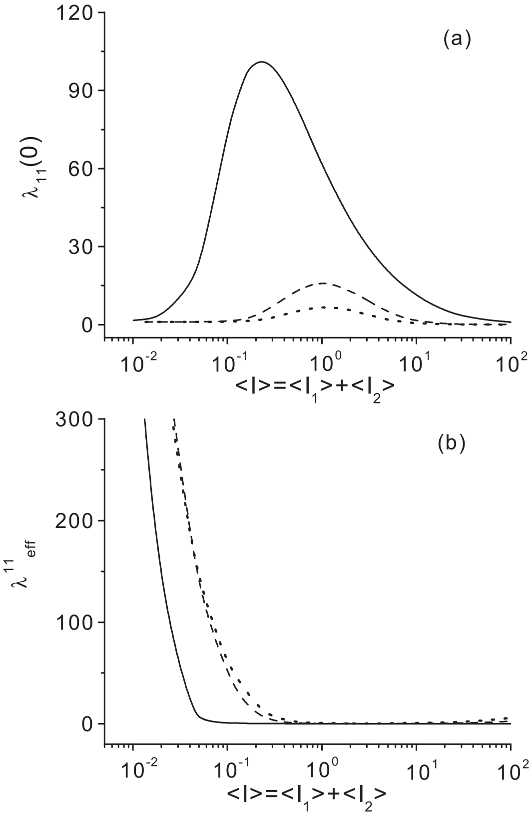

Fig. 5. The intensity variance $\lambda_{11}(0)$ and effective eigenvalue $\lambda^{11}_{\text{eff}}$ of the two-mode dye laser as a function of the average intensity $\langle I\rangle = \langle I_1\rangle + \langle I_2\rangle$ for different values of the noise correlation time τ where $\lambda_{11}(0)$, $\lambda^{11}_{\text{eff}}$, $\langle I\rangle$, and τ are dimensionless. The parameters are dimensionless and chosen as follows: $P' = 100$, $P = 1$, $A = 1$. (—) $\tau = 0.0$; (− − −) $\tau = 0.2$; ($\cdots$) $\tau = 0.5$. (a) $\lambda_{11}(0)$; (b) $\lambda^{11}_{\text{eff}}$.

The theoretical predictions of equations (33, 34) and experimental measurements [22,23] of the variance $\lambda_{11}(0)$ and effective eigenvalue $\lambda^{11}_{\text{eff}}$ of the two-mode dye laser intensities I_1 and I_2 are plotted in Figure 4. Due to the conversion of the experimental data in references [22,23], the theoretical results from equations (33, 34) need to be shifted certain amount to the left. After this correction, good agreement between the decoupling theory and the experimental measurements is obtained.

The intensity fluctuation $\lambda_{11}(0)$ and the eigenvalue $\lambda^{11}_{\text{eff}}$ of the two-mode dye laser are plotted in Figure 5 for different values of the noise correlation time τ. It is clear that the behavior of the curves of $\lambda_{11}(0)$ and $\lambda^{11}_{\text{eff}}$ is similar to that for one-mode dye laser. This means that the noise color can also suppress the fluctuations in a coupled nonlinear dynamical system.

6 Discussion

A two-dimensional decoupling theory is developed when colored noise is included in a nonlinear dynamical system. When the theory is applied to a single mode dye laser system, the laser field can be separated into two parts of amplitude r and phase θ. The variance and the effective eigenvalue λ_{eff} of the laser intensity can be calculated from the steady state distribution function $Q_{\text{s}}(r)$. Also the variance $\lambda_{2\theta}(0)$ and the power spectrum $S(\omega)$ of the laser field can be obtained from the solution of the phase diffusion equation $\Phi(\theta, t)$. When the theory is applied to a two-mode dye laser system with coupling constant $\xi = 2$, the variance $\lambda_{11}(0)$ and the effective eigenvalue $\lambda^{11}_{\text{eff}}$ of the laser intensity I_1 can be calculated from the steady state distribution function $Q_{\text{s}}(r_1, r_2)$ if the mean field theory is employed. It is seen that non-zero noise correlation time τ, namely the noise color, can reduce the fluctuations in both intensity and phase of the laser field. From the solution of the phase diffusion equation (22), it is seen that $\Phi(\theta, t)$ tends to zero as the laser system approaches steady state with $t \to \infty$. This is the reason that only fluctuation of the steady state laser intensity is concerned in most of the theoretical and experimental investigations in laser systems [19,20,22,23].

It should be noted that neglecting terms multiplied by both noises in equation (5), an ensemble average amounts to linearization of the equations for small noises.

Obviously it is not an important approximation since the results agree so well with simulations and also experiment.

The excellent agreement between decoupling theory, numerical simulations and experimental measurements in one-mode and two-mode dye laser systems shows that the decoupling theory is quite successful in dealing with two-dimensional and even high-dimensional coupled nonlinear systems. These systems could be optical systems, electronic and magnetic systems, chemical systems, and neuronal systems, etc. [4].

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